A Heuristic for Multiplicity Calculation in Zaremba's Conjecture

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Continued Fractions

Given $\alpha \in [0, 1]$. To create a continued fraction for α:

1. Let $a_1 = \lfloor \alpha \rfloor$.
2. Form $\alpha_1 = \frac{1}{\alpha} - a_1$. Notice that $\alpha_1 \in [0, 1]$.
3. If $\alpha_1 \neq 0$ let $a_1 = \lfloor \frac{1}{\alpha_1} \rfloor$. Otherwise terminate this process.
4. Form $\alpha_2 = \frac{1}{\alpha_1} - a_2$
5. Repeat the process indefinitely or until termination.

Then

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Each a_i is called a **partial quotient**, and $\alpha = [a_1, ...]$.

Zaremba's Conjecture

Conjecture

There exists A such that for all $q \in \mathbb{N}$ there exists $a \in \mathbb{Z}$ with $(a, q) = 1$ and $\frac{a}{q}$ has partial quotients bounded by A.

Motivation: Quasi-Monte Carlo methods for selecting optimal sampling points for multi-dimensional numerical integration of multivariate integrands.

Matrix Connection to Continued Fractions

If $\alpha = \frac{b}{d} = [a_1, ..., a_n]$

$$\begin{bmatrix} * & b \\ * & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}$$

Let the family of matrices

$$S = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix} \right\}_{i=1}^{A}$$

And form the semi-group with the operation of matrix multiplication

$$\Gamma_A = \langle S \rangle^+ \cap SL_2(\mathbb{Z})$$

Multiplicity in Γ_A

Multiplicity Question: What is the size of the pre-image in Γ_A of some value $n \in \mathbb{N}$ under the projection $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto d$?

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Hardy-Littlewood Approach: Turn a counting question into a contour integral.

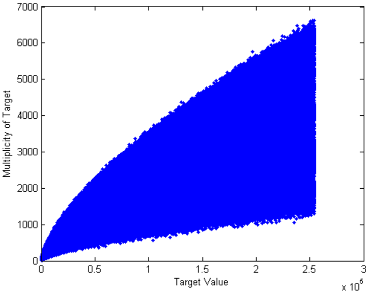
$$mult_N(n) = \sum_{\gamma \in B_N(\Gamma)} \mathbb{1}_{\{f(\gamma)-n=0\}} = \sum_{\gamma \in B_N(\Gamma)} \int_0^1 e\Big(x(f(\gamma) - n)\Big)\, dx$$

Computational results

Simulation in C++ exactly computes the multiplicities for a finite number of elements.

- "Build" the Cayley graph of Γ_A recursively.
  - Only construct exactly as much of the Cayley graph as needed.
- While building the Cayley graph, tally multiplicities.

This algorithm allows us to test very large sets of values extremely efficiently.

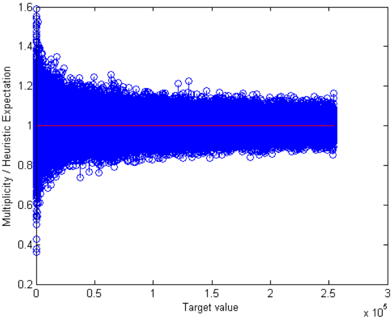


Theoretical Results

Singular Series:

$$mult(n) \sim \frac{2\delta \left| B_n(\Gamma_A) \right|}{n} \mathfrak{S}(n).$$

Where $\mathfrak{S}(n) = \prod_{p|n} \left(\frac{p-1}{p}\right) \zeta(2)$



Future Directions

- Develop similar heuristics for different projections from Γ_A (ex. the trace projection).
- Compute the **major arc** analysis for our current heuristic; thereby conforming to the Hardy-Littlewood Method.
- Approach similar multiplicity problems for different matrix groups (both subgroups of $PSL_2(\mathbb{Z})$ and the Apollonian Group).

Thank you

Thank you Katie, Jen, Li, Lolly, Matt, Max, Senia, and Professor Kontorovich for all of your support and advise.

Additionally, my thanks goes out to the DIMACS/Rutgers Math Department staff and faculty and my follow REU participants.

DIMACS

Center for Discrete Mathematics & Theoretical Computer Science
Founded as a National Science Foundation Science and Technology Center